**Lecture 2: Probability**

1. **Probability: Definition**

Consider an experiment with a finite number of *mutually exclusive* outcomes which are *equiprobable*. Let *A* denote some event associated with the possible outcomes of the experiment.

The **probability** P(*A*) of the event *A* is defined as the fraction of the outcomes in which *A* occurs:

**P(*A*) = N(*A*)/N**,

where N is the total number of outcomes of the experiment and N(*A*) is the number of outcomes leading to the occurrence of the event *A*.

1. **Relative Frequency**

Suppose the experiment can be repeated any number of times, so that we can produce a whole series of *independent trials under identical conditions*, in each of which event A either occurs or does not occur.

Let n be the total number of experiments in the whole series of trials, and let n(*A*) be the number of experiments in which *A* occurs. Then the ratio **n(*A*)/n** is called the **relative frequency** of the event *A* (in the given series of trials).

It turns out that the relative frequencies n(*A*)/n observed in different series of trials are virtually the same for large n, clustering about P(*A*).

1. **Elementary Events, Complementary Event**

The mutually exclusive outcomes of a random experiment are called **elementary events**. A typical elementary event will be denoted by ω.

The set of all elementary events ω associated with a given experiment is called the **sample space**, denoted by *Ω*. An event *A* is said to be *associated with the elementary events of Ω* if, given any ω in *Ω*, we can always decide whether or not ω leads to the occurrence of *A*.

The same symbol *A* will be used to denote both the event *A* and the set of elementary events leading to the occurrence of *A*. An event *A* occurs if and only if one of the elementary events ω in the set *A* occurs.

For an event *A*, the event “*A* does not occur” is called the **complementary event** of *A*, denoted by *Ā*.

1. **Independent Events and Mutually Exclusive Events**

Two events *A1* and *A2* are said to be **independent** if the occurrence of one event has no influence on the probability of the occurrence of the other.

Two events *A1* and *A2* are said to be **mutually exclusive** or **incompatible** if the occurrence of one event precludes the occurrence of the other, i.e., if *A1* and *A2* cannot occur simultaneously.

1. **Multiplication Rule**

If two events *A1* and *A2* are independent,

**P(*A1***[∩](http://www.cut-the-knot.org/do_you_know/add_set.shtml#intersect) ***A2*) =P(*A1*) × P(*A2*)** .

*Generalization*: If the events *A1*, *A2* , … , *Ak* are all independent of each other,

**P(*A1***[∩](http://www.cut-the-knot.org/do_you_know/add_set.shtml#intersect) ***A2*** [∩](http://www.cut-the-knot.org/do_you_know/add_set.shtml#intersect) ***…*** [∩](http://www.cut-the-knot.org/do_you_know/add_set.shtml#intersect) ***Ak*) =P(*A1*) × P(*A2*) × … × P(*Ak*)** .

1. **Addition Rule**

If two events *A1* and *A2* are mutually exclusive,

**P( *A1***[∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) ***A2* ) = P(*A1*) + P(*A2*)** .

*Generalization*: If *Ai* and *Aj* are mutually exclusive for any *i* and *j*,

**P( *A1***[∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) ***A2*** [∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) **…** [∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) ***Ak* ) = P(*A1*) + P(*A2*) + … + P(*Ak*)**.

1. **Addition Rule (Cont’d)**

Since *A* and *Ā* are mutually exclusive, P( *A*[∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) *Ā* ) = P(*A*) + P(*Ā*) . But

P( *A*[∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) *Ā* ) = P(*Ω*) = 1, from which we have P(*Ā*) = 1 – P(*A*) .

Also note that if *A1* and *A2* are independent, *A1* and *Ā2* are also independent, and

P(*Ā1*[∩](http://www.cut-the-knot.org/do_you_know/add_set.shtml#intersect) *Ā2*) =P(*Ā1*) × P(*Ā2*) =( 1 *–* P(*A1*)) × ( 1 – P(*A2*))

=1 *–* P(*A1*) – P(*A2*) + (P(*A1*) × P(*A2*)) =1 *–* P(*A1*) – P(*A2*) + P(*A1* [∩](http://www.cut-the-knot.org/do_you_know/add_set.shtml#intersect) *A2*)

=1 *–* P(*A1* [∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) *A2*) .

Therefore,

**P( *A1***[∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) ***A2* ) = 1 *–* P(*Ā1***[∩](http://www.cut-the-knot.org/do_you_know/add_set.shtml#intersect) ***Ā2*) = 1 *–* P(*Ā1*) × P(*Ā2*)** .

1. **Problems**
2. There are 12 girls and 8 boys in a class of 20. If a student is chosen at random, what is the probability that a boy is chosen?
3. On every school day the students draw lots to decide who will be on duty that day. In a week of 5 school days, what is the probability that all 5 students on duty are girls?
4. If a group of 5 is chosen from the above class, what is the probability that the group consists of girls only?
5. What is the probability that there is at least one boy in the group chosen above?
6. **Problems**
7. If 10 students are chosen at random from the class, what is the probability that exactly 6 are girls?
8. If the students draw lots on every school day to decide who will be on duty that day, in a week of 5 school days, what is the probability that

(a) 3 girls and 2 boys are on duty,

(b) no more than 3 girls are on duty,

(c) no student is on duty for more than one day?

1. **Problems**
2. If each one of 20 students chooses at random a number from 1 to 100, what is the probability that the 20 chosen numbers are all different?
3. If the exam scripts of 20 students are distributed at random among the students, what is the probability that no one gets his/her own script?
4. **Conditional Probability**

The **conditional probability of *A* on the hypothesis *B***, i.e., the probability of *A* occurring under the condition that *B* is known to have occurred, is defined by

**P(*A* | *B*) = P(*A***[**∩**](http://www.cut-the-knot.org/do_you_know/add_set.shtml#intersect) ***B*) / P(*B*) .**

As P(*A*[∩](http://www.cut-the-knot.org/do_you_know/add_set.shtml#intersect) *B*) = P(*A*|*B*) × P(*B*) = P(*B*|*A*) × P(*A*) , we have

**P(*B* | *A*) = P(*A* | *B*) × P(*B*) / P(*A*)** .

Suppose *B1*, *B2*, …, *Bk* is a “full set” of mutually exclusive events, i.e., *B1*[∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) *B2* [∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) … [∪](http://www.cut-the-knot.org/do_you_know/add_set.shtml#union) *Bk* = *Ω .*

Then we have the **total probability formula**:

**P(*A*) = P(*A*|*B1*) × P(*B1*) + P(*A*|*B2*) × P(*B2*) + … + P(*A*|*Bk*) × P(*Bk*) .**

**Bayes’ rule:** If *B1*, *B2*, …, *Bk* is a full set of mutually exclusive events, then

**P(*Bk*|*A*) = P(*A*|*Bk*) × P(*Bk*) / P(*A*)**

**= P(*A*|*Bk*) × P(*Bk*) / ( P(*A*|*B1*) × P(*B1*) + P(*A*|*B2*) × P(*B2*) + … + P(*A*|*Bk*) × P(*Bk*)) .**

1. **Problems**
2. In a class of 12 girls and 8 boys, 7 of the girls and 5 of the boys are Chinese. All the others are not. If a Chinese student is chosen at random, what is the probability that a girl is chosen?
3. If a girl student is chosen at random from a class, the probability that she is Chinese is 60%. If a boy is chosen, the probability that he is Chinese is 70%. There are twice as many girls as boys in the class. If a Chinese student is chosen at random, what is the probability that a girl is chosen?
4. **Monty Hall Problem**

A guest X goes on a game show. He is given the choice of three doors. Behind one door is a car; behind the two others, goats. X picks a door, say No. 1, and the host H, who knows what is behind each door, opens another door, say No. 3, which has a goat. H then says to X, "Do you want to switch your choice to door No. 2?"

Is it to X’s advantage to switch his choice?